CALCULATION OF THE ONSET OF TURBULENCE IN NON-NEWTONIAN FLUID FLOW

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The onset of the transition regime in a Newtonian fluid flow is determined by the critical value of the Reynolds number, below which turbulent motion is impossible. Several modifications of the Reynolds number have been proposed for non-Newtonian fluids [1-8].

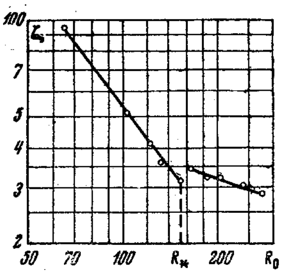

Fig. 1. Resistance coefficient ζ as a function of R_0 number for a 0.3% aqueous solution of carbopol, D = 2.54 cm.

The most widely used criterion is that introduced by Metzner and Reed [4]. On the basis of an examination of the equations of motion and continuity Bird [5] has shown that this criterion cannot uniquely characterize the moment of onset of the transition regime, as has since been experimentally confirmed by Ulbrecht [7]. Bird's remark also applies to the criterion used by Ulbrecht. As noted in [7], this criterion was observed to be constant in the range of values of the exponent $0.6 < n < 1.0$, i.e., for fluids with relatively weakly expressed non-Newtonian properties.

Ryan and Johnson [6] have proposed a generalized local Reynolds number of the type:

$$Z = \frac{D\rho w}{\tau_w}\frac{\partial w}{\partial y}.$$

Here, D is the diameter, ρ is the density, w is the local velocity, τ_w is the shear stress at the wall, and y is the coordinate normal to the wall.

Since the expression for Z contains local values of the velocity and velocity gradient, it varies over the channel cross section. Ryan and Johnson have suggested that the maximum value of Z is the same for both Newtonian and non-Newtonian fluids.

We transform the criterion Z as follows:

$$Z = R\omega\sigma \quad (R = \rho\varphi D\langle w\rangle,\ \omega = w/\langle w\rangle,\ \sigma = \tau/\tau_w).$$

Here, R is the Reynolds number, φ is the intrinsic fluidity, $\langle w\rangle$ is the mean velocity, and τ is the local value of the shear stress.

From this expression for Z it is immediately clear that it cannot be the same for Newtonian and non-Newtonian fluids if only because the relative velocities ω are described by different expressions [9].

The same remark also applies to the criterion proposed by Hanks [8], which for a circular pipe differs from Z by a constant factor.

1. Formulation of the problem. We will consider the equation of motion

$$\rho\partial w_i/\partial t + \rho w_j\partial w_i/\partial x_j = -\partial p/\partial x_i + \partial\tau_{ji}/\partial x_j \quad (i, j = 1, 2, 3). \quad (1.1)$$

Here, w_i are velocity vector components, t is time, p is pressure, τ_{ji} are the stress deviator components, and x_j are the coordinates of a rectangular Cartesian coordinate system. We introduce the dimensionless variables.

$$\xi = x/l,\quad \omega = w/w_0,\quad \vartheta = w_0 t/l,\quad \pi = p/\rho w_0^2,\quad \sigma = \tau/\tau_w. \quad (1.2)$$

Here, w_0 and l are the characteristic velocity and length, respectively.

The equation of motion takes the form

$$\frac{\partial\omega_i}{\partial\vartheta} + \omega_j\frac{\partial\omega}{\partial\xi_j} = -\frac{\partial\pi}{\partial\xi_i} + \frac{\zeta}{8}\frac{\partial\sigma_{ji}}{\sigma\xi_j} \qquad \left(\zeta = \frac{s\tau_w}{\rho w_0^2}\right), \quad (1.3)$$

where ζ is the resistance coefficient.

We define a fluid with structural viscosity as a fluid whose physical parameters do not depend on time. A Newtonian fluid is a particular case of a fluid with structural viscosity, when the viscosity does not depend on the shear stress.

Since Eq. (1.3) uniquely describes the process of motion of the fluid and the only integral characteristic of this equation is the resistance coefficient ζ, it is natural to assume that in the flow of fluids with structural viscosity the onset of the transition regime must correspond to the critical value of the resistance coefficient ζ_*. The constancy of ζ_* was reported in [4] on the basis of a large number of experimental data.

Here, it should be noted that we are considering fluids with zero yield stress, i.e., fluids that begin to flow even upon the application of an infinitely small load. If the fluid has a yield point, then a solid core develops at the center of the flow, and Eq. (1.3) is not applicable to the entire cross section of the channel.

2. Circular pipes. For a Newtonian fluid flowing in a circular pipe $\zeta_* \approx 2.8\cdot 10^{-2}$.

Following [9], we assume that the motion of a fluid with structural viscosity obeys Newton's law of viscous friction with variable fluidity:

$$\varphi\boldsymbol{\tau} = (\mathbf{n}\cdot\nabla)\,\mathbf{w}. \quad (2.1)$$

We employ the linear law [9]

$$\varphi = \varphi_0 + \theta\,|\tau. \quad (2.2)$$

Here, φ_0 is the fluidity at zero shear and θ is the stability coefficient.

In polar coordinates with the center on the axis of the pipe we have the relation

$$\sigma = \xi\ (\xi = r/r_0). \quad (2.3)$$

Here, r is the radial coordinate and r_0 is the radius of the pipe.

Comparing Eqs. (2.1) and (2.2), we find

$$dw/dr = -\varphi_0\tau - \theta\tau^2. \quad (2.4)$$

Using the dimensionless variables (1.2) and relation (2.3), we transform Eq. (2.4):

$$\frac{d\omega}{d\xi} = -\frac{R_0\zeta}{16}\xi - \frac{R_0^3\zeta^2}{128\theta_*}\xi^2$$

$$\left(R_0 = \rho\varphi_0 D\langle w\rangle,\ \theta_* = \frac{\rho\varphi_0^3 D^2}{\theta},\ \zeta = \frac{8\tau_w}{\rho\langle w\rangle^2}\right). \quad (2.5)$$

We write the continuity equations in the form

$$2\int_0^1 \xi\omega\, d\xi = 1. \quad (2.6)$$

Integrating by parts, we find

$$\int_0^1 \xi^2\frac{d\omega}{d\xi}d\xi = -1. \quad (2.7)$$

Substituting (2.5) into (2.7) and setting $\zeta = \zeta_*$, we obtain

$$R_*^3 + 3.6\cdot 10^2\,\theta_* R_* - 8.2\cdot 10^5\,\theta_* = 0. \quad (2.8)$$

Here, R_* is the critical value of the R_0 number, which corresponds to the beginning of the transition regime.

We write the expression for R_* in explicit form using the Cardan formula:

$$R_* = (4.1\cdot 10^5\theta_* + \sqrt{1.68\cdot 10^{11}\theta_*^2 + 2.2\cdot 10^6\theta_*^3})^{1/3} + (4.1\cdot 10^5\theta_* - \sqrt{1.68\cdot 10^{11}\theta_*^2 + 2.2\cdot 10^6\theta_*^3})^{1/3}. \quad (2.9)$$

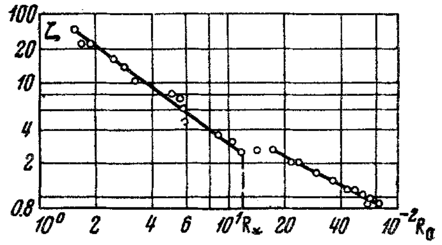

Fig. 2. Resistance coefficient ζ as a function of R_0 number for a 0.3% aequeous solution of type H natrosol, D = 2.54 cm.

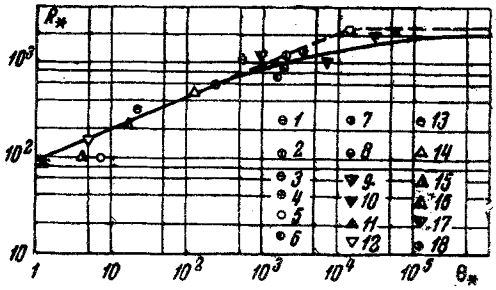

Fig. 3. Critical Reynolds number R_* as a function of the criterion θ_*. The points 1–18 correspond to different aqueous solutions; the data have been taken from the references indicated in brackets; values of the diameter in centimeters are also given:

1	0.3%	natrosol type H	1.27	[12]
2	0.3%	natrosol type H	2.54	[12]
3	1.0%	natrosol type G	1.27	[12]
4	1.0%	natrosol type G	2.54	[12]
5	0.29%	ammonium oleate	0.30	[13]
6	0.75%	carboxymethyl cellulose	1.47	[6]
7	0.75%	carboxymethyl cellulose	1.90	[6]
8	1.0%	carboxymethyl cellulose	1.27	[6]
9	0.2%	carbopol	1.27	[6]
10	0.2%	carbopol	1.54	[6]
11	0.3%	carbopol	1.27	[6]
12	0.3%	carbopol	2.54	[6]
13	0.4%	carbopol	1.27	[6]
14	0.4%	carbopol	2.54	[6]
15	0.5%	carbopol	1.27	[6]
16	0.5%	carbopol	2.54	[6]
17		butadiene-styrene latex № 3	1.975	[7]
18		butadiene-styrene latex № 4	1.975	[7]

Expression (2. 9) can be approximated as follows:

$$R_* \approx 93.5\,(\theta_*)^{1/3} \quad \text{at} : \theta_* < 1.5\cdot 10^4,$$
$$R_* \approx 2300 \quad \text{at} \quad \theta_* \geqslant 1.5\cdot 10^4\,. \tag{2. 10}$$

3. **Parallel plates.** For a fluid with fluidity given by (2. 2) the dimensionless velocity gradient has the form

$$\frac{d\omega}{d\xi} = -\frac{R_0\zeta}{32}\,\xi - \frac{R_0^3\zeta^2}{256\theta_*}\,\xi^2,$$
$$R_0 = 4\rho\varphi_0 h\,\langle w\rangle, \quad \theta_* = \frac{16\rho\varphi_0^3 h}{\theta},$$
$$\xi = \frac{y}{h}. \tag{3. 1}$$

Here, 2h is the distance between plates.

We write the continuity equations as

$$\int_0^1 \omega\, d\xi = 1. \tag{3. 2}$$

On the basis of [10], in which the transition regime was investigated for water flowing in a rectangular channel with a side ratio of 1:20, we can set $\zeta_* = 3.6\cdot 10^{-2}$.

Substituting this value of the resistance coefficient in (3. 1) and jointly solving (3. 1) and (3. 2), we obtain

$$R_*^3 + 3\cdot 10^3\,\theta_* R_* - 7.9\cdot 10^5\,\theta_* = 0. \tag{3. 3}$$

It is also possible to use the approximate relations

$$R_* \approx 92.5\sqrt[3]{\theta_*} \quad \text{at} \quad \theta_* < 2.5\cdot 10^4\,,$$
$$R_* \approx 2700 \quad \text{at} \quad \theta_* \geqslant 2.5\cdot 10^4\,. \tag{3. 4}$$

4. **Annular gap.** For a fluid with structural viscosity flowing in the axial direction between concentric cylinders the dimensionless velocity gradient is

$$\frac{d\omega}{d\xi} = -\frac{R_0\xi}{16\,(1-\xi_a^2)}\left(\xi - \frac{\xi_0^2}{\xi}\right) -$$
$$-\frac{R_0^3\zeta^2}{128\theta_*\,(1-\xi_a^2)}\left(\xi^2 - 2\xi_a^2 + \frac{\xi_a^4}{\xi^2}\right),$$
$$R_0 = 2\rho\varphi_0\,(r - r_0)\,\langle w\rangle, \qquad \theta_* = \frac{4\rho\varphi_0^3\,(r-r_0)^2}{\theta},$$
$$\xi = \frac{r}{r_0}, \qquad \xi_a = \frac{r_a}{r_0}, \qquad \xi_a = \frac{-b + \sqrt{b^2 + 4ac}}{2c},$$
$$a = \frac{R_0\zeta\ln\xi_0}{16\,(1-\xi_0)} - \frac{R_0^3\zeta^2}{128\theta_*\xi_0},$$
$$b = \frac{R_0\zeta\,(1 - \xi_0^2 - 2\ln\xi_0)}{32\,(1-\xi_0)} + \frac{R_0^3\zeta^2}{64\theta_*},$$
$$c = \frac{R_0\zeta\,(1+\xi_0)}{32} + \frac{R_0^3\zeta^2\,(1+\xi_0+\xi_0^2)}{384\theta_*}, \qquad \xi_0 = \frac{r_0}{r}. \tag{4. 1}$$

Here r_a is the radius corresponding to the maximum velocity and r_0 and r are the radii of the inside and outside cylinders, respectively.

After integration by parts the continuity equation takes the form

$$\int_0^1 \xi^2\,\frac{d\omega}{d\xi}\,d\xi = \xi_0^2 - 1\,. \tag{4. 2}$$

Solving (4. 1) and (4. 2) jointly, we find that R_* is given by the nonlinear equation

$$\frac{R_*^3\zeta_*^2}{128\theta_*\,(1-\xi_a^2)^2}\left[\frac{1-\xi_0^5}{5} - \frac{2}{3}\,\xi_a^2\,(1-\xi_0^3) + \xi_a^4\,(1-\xi_0)\right] +$$
$$+\frac{R_*\zeta_*}{16\,(1-\xi_a^2)}\left[\frac{1-\xi_0^4}{4} - \frac{\xi_a^2}{2}\,(1-\xi_a^2)\right] -$$
$$-(1-\xi_0)^2\,(1+\xi_0) = 0\,. \tag{4. 3}$$

The critical value of the resistance coefficient varies as a function of the ratio of the radii ξ_0. Since reliable experimental data over the entire range of values of ξ_0 are not available, it is recommended to use the value of ζ_* for a circular pipe at ξ_0 close to zero and the ζ_* for parallel plates at ξ_0 close to unity.

5. **Boundary layer on a plate.** We write the dimensionless velocity in the form

$$\omega = 4.25 R_0^{**} c_f\,(\xi - \xi^3 + {}^1\!/_2\,\xi^4) +$$
$$2.13\theta_*^{-1}\, R_0^{**3} c_f^2\,({}^4\!/_7\,\xi + 2\xi^6 + {}^9\!/_5\,\xi^5 + \xi^4 - 2\xi^3 + \xi)\,,$$
$$R_0^{**} = \rho\varphi_0\delta^{**} w_0, \quad \theta_* = \frac{\rho\varphi_0^3\delta^{**2}}{\theta}, \quad c_f = \frac{2\tau_w}{\rho w_0^2},$$
$$\omega = \frac{w}{\langle w\rangle}, \quad \xi = \frac{0.179y}{\delta^{**}}. \tag{5. 1}$$

Here, c_f is the friction coefficient, δ^{**} is the momentum thickness, and w_0 the potential flow velocity. At $\omega = 1$ and $\xi = 1$

$$R_*^{**3} + \frac{2.69\theta_*}{c_{f*}}\,R_*^{**} - 1.26\,\frac{\theta_*}{c_{f*}} = 0. \tag{5. 2}$$

Onset of the transition regime in the boundary layer on a plate depends importantly on the intensity of turbulence in the external flow. The friction coefficient for a Newtonian fluid can be found from the relation $c_f = 0.44/R^{**}$ and the curve expressing the critical Reynolds number [12] as a function of $(u_1^2 + u_2^2 + u_3^2)/3w_0^2$, where u_i are the free-stream fluctuation velocity components.

6. **Comparison with experiment.** The experimental data of [6, 7, 12, 13] on the flow of pseudo-plastic fluids ($\theta > 0$) in circular pipes have been analyzed in the form of relations $\varphi_* - \varphi(\tau)$ and $\zeta = \zeta(R_0)$.

These fluids approximately obey a linear fluidity law up to the onset of the transition regime. The constants of this law—the fluidity at zero shear φ_0 and the stability coefficient θ—were found by the method of least squares. The beginning of the transition regime was determined from the break in the $\zeta = \zeta(R_0)$ curve. Typical graphs are presented in Figs. 1 and 2.

In Fig. 3 the experimental data are compared with formulas (2. 9) (solid line) and (2.10) (dashed line). The agreement with experiment may be considered adequate.

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